

Farrell-Jones via Dehn fillings.

Yago Antolín, Rémi Coulon, Giovanni Gandini

Abstract

Following the approach of Dahmani, Guirardel and Osin, we extend the group theoretical Dehn filling theorem to show that the pre-images of infinite order elements have a certain structure of a free product. We then apply this result to show that groups hyperbolic relative to residually finite groups satisfying the Farrell-Jones conjecture, they satisfy the Farrell-Jones conjecture.

1 Introduction

The *Farrell-Jones conjecture* aims to describe the algebraic K -theory of a group ring in terms of the algebraic K -theories of simpler group rings. This is formulated by claiming that a certain map is an isomorphism [12]. Bartels, Lück and Reich introduced a series of isomorphism conjectures inspired by the insight of Farrell and Jones, that are more general and have stronger implications and inheritance properties. Since many geometric obstructions live in the algebraic K -theory of the group ring, the validity of the Farrell-Jones conjecture, has several outstanding consequences, for example the *Novikov conjecture*, the *Borel conjecture*, the *Kaplansky conjecture*, and the *Serre conjecture* [19]. In the last 10 years that has been a tremendous amount of work devoted in solving the Farrell-Jones conjecture for several classes of groups, and often its solution gave the first solution to some of the famous conjectures above [19, 6, 5, 24, 25, 4].

Now we start by describing the statement of the version of the Farrell-Jones conjecture that we consider here. For a group G , let \underline{EG} be the classifying space of G for the family of virtually cyclic subgroups. We recall that a classifying space of G for the family of virtually cyclic subgroups is a G -CW-complex X with the property that the fixed point subcomplex X^H is contractible for every virtually cyclic subgroup H of G and empty otherwise. This space is generally a rather unusual G -space, some examples can be found in [18]. Now, let \mathcal{A} be an additive G -category, the projection $\underline{EG} \rightarrow \text{pt}$ induces an “assembly map”

$$\text{asmb}_n^{G, \mathcal{A}} : H_n^G(\underline{EG}; \mathbf{S}) \rightarrow H_n^G(\text{pt}; \mathbf{S}),$$

where $H_n^G(\underline{EG}; \mathbf{S})$ is a G -equivariant homology theory, and \mathbf{S} can be either the $\text{Or}G$ -spectrum $\mathbf{K}_{\mathcal{A}}$ or the $\text{Or}G$ -spectrum $\mathbf{L}_{\mathcal{A}}$, please consult [8] for various definitions and details. Here we say that a group G satisfies the *Full Farrell-Jones Conjecture* if for any finite group F the assembly map for the wreath product $G \wr F$ is an isomorphism in any additive G -category. When \mathcal{A} is the category of finitely generated free modules over some ring R with trivial G -action, then this reduces to say that the map $H_n^G(\underline{EG}; \mathbf{K}_{\mathbf{R}}) \rightarrow K_n(RG)$, is an isomorphism. In other words, the algebraic K -theory $K_n(RG)$ of the group ring RG can be constructed from the algebraic K -theories $K_n(RH)$ ’s where H ranges over the family of virtually cyclic subgroups.

Recently, Bartels, using a new notion of “coarse flow space” solved the Farrell-Jones conjecture for all hyperbolic groups relative to a family of subgroups satisfying the Farrell-Jones conjecture [3]. Bartels’ proof uses a far reaching generalisation of the *geodesic flow method* introduced in [12] which was also adapted in the solution of the Farrell-Jones conjecture for hyperbolic groups [6]. We extend the Dehn filling Theorem following Dahmani, Guirardel, and Osin [11] and combine it with the solution of the Farrell-Jones conjecture for hyperbolic groups to give an alternative proof of the conjecture for relative hyperbolic groups when the peripheral subgroups are residually finite in addition to satisfy the Farrell-Jones conjecture. Note that this case covers the geometrically relevant examples of relatively hyperbolic groups, such as fundamental groups of complete hyperbolic manifolds of finite volume and fundamental groups of complete Riemannian manifolds of finite volume with pinched negative sectional curvature. Our proof relies on the strong inheritance properties of the (Full) Farrell-Jones conjecture, Lück-Bartels-Reich solution for hyperbolic groups, and a detailed study of the group structure of relatively hyperbolic groups. The conjecture remains unknown for many groups connected to this work, such as the outer automorphism groups of a right-angled Artin group or even for $Out(F_n)$, mapping class groups and more generally for *acylindrically hyperbolic groups*.

Dehn fillings or Dehn surgery is a powerful tool to produce quotient groups and spaces of negative curvature. Group theoretical Dehn fillings were inspired by Thurston’s hyperbolic Dehn surgery Theorem [23] which says that if M is an hyperbolic 3-manifold with a single torus cusp C , then for all but finitely many $g \in \pi_1(C)$ the quotient group $\pi_1(M)/\langle\langle g \rangle\rangle$ is infinite, non-elementary and word hyperbolic. The group theoretical version of this theorem for relatively hyperbolic groups is due to Groves and Manning [16] and Osin [21]. The theorem has been further generalized to the context of acylindrically hyperbolic groups by Dahmani, Guirardel and Osin [11]. In cite [11], using Gromov’s rotating families [15] and windmills, the authors are able to describe the kernel of a Dehn filling and they show that it isomorphic to a free product.

The strategy we follow to prove the Farrell-Jones conjecture is based on the stability of the conjecture under certain type of group extensions. More concretely, if G is an extension of groups satisfying the Farrell-Jones conjecture, then G itself satisfies the Farrell-Jones conjecture provided that the preimage in G of any infinite cyclic subgroup of the quotient satisfies the conjecture. Then, if one starts with a relatively hyperbolic group G with residually finite parabolic subgroups satisfying the Farrell-Jones conjecture, one can use the Dehn fillings theorem to obtain a short exact sequence $K \rightarrow G \xrightarrow{\pi} Q$, where K and Q satisfy the Farrell-Jones conjecture and the problem relies on understanding $\pi^{-1}(\langle q \rangle)$ for $q \in Q$ of infinite order. Our main technical contribution is item (iii) of the theorem below.

Theorem 1.1. *Let G be finitely generated group hyperbolic relatively to a family of subgroups $\{P_1, \dots, P_n\}$. There is a finite set $\Phi \subseteq G \setminus \{1\}$ such that whenever we take finite index normal subgroups $N_i \trianglelefteq P_i$ with $N_i \cap \Phi = \emptyset$ for $i = 1, \dots, n$ then the following hold:*

- (i) $\bar{G} := G/K$ is an hyperbolic group where K is the normal subgroup of G generated by $N_1 \cup \dots \cup N_n$.
- (ii) there exists subsets T_i of G for $i = 1, \dots, n$ such that K is isomorphic to $*_{i=1}^n (*_{t \in T_i} N_i^t)$.
- (iii) for every $\bar{g} \in \bar{G}$ of infinite order, there is a pre-image g of \bar{g} under the natural map $G \rightarrow \bar{G}$

and subsets T'_i of G for $i = 1, \dots, n$ such that

$$\langle g, K \rangle = \langle g \rangle * \left[*_{i=1}^n \left(*_{t \in T'_i} N_i^t \right) \right].$$

Note that (i) already appears in [21] and (ii) appears in [11]. Our proof of (iii) follows the strategy of the one of (ii) of Dahmani, Guirardel and Osin, and for that, we introduce a variation of the windmills used in [11].

The group theoretical Dehn fillings Theorem (and variations) have proved to be a extremely useful tool in modern geometric group theory. There is a good number of interesting applications: it has been used to construct simple groups with arbitrarily large ℓ^2 -Betti number [22], to prove that normal automorphisms of acylindrically hyperbolic groups with trivial finite radical are inner [2], and it plays an important role in the solution of the Virtual Haken conjecture [1]. Therefore, Theorem 1.1 (iii) is interesting not only for its application to the Farrell-Jones conjecture, but also for obtaining a better understanding of the Dehn fillings theorem itself.

The structure of the paper is as follows: in Section 2 we set notation and review all the basic definitions about hyperbolic geometry. Section 3 is the core of the paper, we introduce the extended windmills and we prove Theorem 3.17 which is a version of Theorem 1.1 for groups acting on hyperbolic spaces. In Section 4, we collect the needed references to deduce Theorem 1.1 from the results of Section 3 and finally in Section 5 we get the main application of the paper, namely Theorem 5.2 where we get our version of the Farrell-Jones conjecture for relatively hyperbolic groups.

Acknowledgments. As we start thinking about this question Bartels' general solution had not appeared. The authors are thankful to D. Osin and A. Bartels who encouraged us though to write down this alternative approach. The authors would like to thank the organizers of the Ventotene International Workshops (2015) where part of this paper was written. The first author acknowledges partial support from the Spanish Government through grant number MTM2014-54896-P. The third author was supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.

2 Hyperbolic geometry

Notations and vocabulary. Let X be a metric length space. Given two points x and x' of X , we denote by $|x - x'|_X$ (or simply $|x - x'|$) the distance between them. Let Y be a subset of X . We write $d(x, Y)$ for the distance between a point $x \in X$ and Y . We write $B(x, r)$ for the closed ball of center x and radius r .

The four point inequality. The Gromov product of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2} \{ |x - z| + |y - z| - |x - y| \}.$$

For the remainder of this section, we assume that the space X is δ -hyperbolic, i.e. for every $x, y, z, t \in X$,

$$\langle x, z \rangle_t \geq \min \{ \langle x, y \rangle_t, \langle y, z \rangle_t \} - \delta, \quad (1)$$

or equivalently

$$|x - z| + |y - t| \leq \max \{|x - y| + |z - t|, |x - t| + |y - z|\} + 2\delta. \quad (2)$$

Remark. If X is 0-hyperbolic, then it can be isometrically embedded in an \mathbf{R} -tree, [14, Chapitre 2, Proposition 6]. For our purpose though, we will always assume that the hyperbolicity constant δ is positive. Indeed, every 0-hyperbolic space is δ -hyperbolic for every $\delta \geq 0$.

It is known that triangles in a geodesic hyperbolic space are 4δ -thin (every side lies in the 4δ -neighborhood of the union of the two other ones). This can be stated through the following metric inequality. In this statement the Gromov product $\langle x, z \rangle_s$ should be thought as a very small quantity. For every $x, y, z, s \in X$,

$$\langle x, y \rangle_s \leq \max \{|x - s| - \langle y, z \rangle_x, \langle x, z \rangle_s\} + \delta. \quad (3)$$

The boundary at infinity. Let x be a base point of X . A sequence (y_n) of points of X *converges to infinity* if $\langle y_n, y_m \rangle_x$ tends to infinity as n and m approach to infinity. The set \mathcal{S} of such sequences is endowed with a binary relation defined as follows. Two sequences (y_n) and (z_n) are related if

$$\lim_{n \rightarrow +\infty} \langle y_n, z_n \rangle_x = +\infty.$$

It follows from (1) that this relation is actually an equivalence relation. The *boundary at infinity* of X , denoted by ∂X , is the quotient of \mathcal{S} by this relation. If the sequence (y_n) is an element in the class of $\xi \in \partial X$, we say that (y_n) *converges* to ξ and write

$$\lim_{n \rightarrow +\infty} y_n = \xi.$$

Note that the definition of ∂X does not depend on the base point x .

Quasi-geodesics. In this article, unless otherwise stated a path is always a rectifiable path parametrized by arc length.

Definition 2.1. Let $l \geq 0$, $k \geq 1$ and $L \geq 0$. Let $f: X_1 \rightarrow X_2$ be a map between two metric spaces X_1 and X_2 . We say that f is a (k, l) -*quasi-isometric embedding* if for every $x, x' \in X_1$,

$$k^{-1} |f(x) - f(x')| - l \leq |x - x'| \leq k |f(x) - f(x')| + l.$$

We say that f is an L -*local* (k, l) -*quasi-isometric embedding* if its restriction to any subset of diameter at most L is a (k, l) -quasi-isometric embedding. Let I be an interval of \mathbf{R} . A path $\gamma: I \rightarrow X$ that is a (k, l) -quasi-isometric embedding is called a (k, l) -*quasi-geodesic*. Similarly, we define L -*local* (k, l) -*quasi-geodesics*.

Remarks. We assumed that our paths are rectifiable and parametrized by arc length. Thus a (k, l) -quasi-geodesic $\gamma: I \rightarrow X$ satisfies a more accurate property: for every $t, t' \in I$,

$$|\gamma(t) - \gamma(t')| \leq |t - t'| \leq k |\gamma(t) - \gamma(t')| + l.$$

In particular, if γ is a $(1, l)$ -quasi-geodesic, then for every $t, t', s \in I$ with $t \leq s \leq t'$, we have $\langle \gamma(t), \gamma(t') \rangle_{\gamma(s)} \leq l/2$. Since X is a length space for every $x, x' \in X$, for every $l > 0$, there exists a $(1, l)$ -quasi-geodesic joining x and x' .

Let $\gamma: \mathbf{R}_+ \rightarrow X$ be a (k, l) -quasi-geodesic. There exists a point $\xi \in \partial X$ such that for every sequence (t_n) diverging to infinity, $\lim_{n \rightarrow +\infty} \gamma(t_n) = \xi$. In this situation we consider ξ as an endpoint (at infinity) of γ and write $\lim_{t \rightarrow +\infty} \gamma(t) = \xi$. In this article we are mostly using L -local $(1, l)$ -quasi-geodesics. Thus we state the stability of quasi-geodesics for this kind of paths.

Corollary 2.2. [10, Corollaries 2.6 and 2.7] *Let $l_0 \geq 0$. There exists $L = L(l_0, \delta)$ which only depends on δ and l_0 such that for every $l \in [0, l_0]$, and every L -local $(1, l)$ -quasi-geodesic $\gamma: I \rightarrow X$, the following hold:*

- (i) *the path γ is a (global) $(2, l)$ -quasi-geodesic,*
- (ii) *for every $t, t', s \in I$ with $t \leq s \leq t'$, we have $\langle \gamma(t), \gamma(t') \rangle_{\gamma(s)} \leq l/2 + 5\delta$,*
- (iii) *for every $x \in X$, for every y, y' lying on γ , we have $d(x, \gamma) \leq \langle y, y' \rangle_x + l + 8\delta$.*
- (iv) *the Hausdorff distance between γ and any other L -local $(1, l)$ -quasi-geodesic joining the same endpoints (possibly in ∂X) is at most $2l + 5\delta$.*

Remark. Using a rescaling argument, one can see that the best value for the parameter $L = L(l, \delta)$ satisfies the following property: for all $l, \delta \geq 0$ and $\lambda > 0$, $L(\lambda l, \lambda \delta) = \lambda L(l, \delta)$. This allows us to define a parameter L_S that will be use all the way through.

Definition 2.3. Let $L(l, \delta)$ be the infimum value for the parameter L given in Corollary 2.2. We denote by L_S a number larger than 500 such that $L(10^5 \delta, \delta) < L_S \delta$.

The stability of quasi-geodesics (Corollary 2.2) has a discrete analogue that we state below.

Proposition 2.4 (Stability of discrete quasi-geodesics). [10, Proposition 2.9] *Let $l > 0$. There exists $L = L(l, \delta)$ which only depends on δ and l such that for every sequence of points x_0, \dots, x_m in X , satisfying that*

- (i) *for every $i \in \{1, \dots, m-1\}$, $\langle x_{i-1}, x_{i+1} \rangle_{x_i} \leq l$,*
- (ii) *for every $i \in \{1, \dots, m-2\}$, $|x_{i+1} - x_i| \geq L$.*

Then for all $i \in \{0, \dots, m\}$, the inequality $\langle x_0, x_m \rangle_{x_i} \leq l + 5\delta$ holds. Moreover, for all $p \in X$ there exists $i \in \{0, \dots, m-1\}$ such that $\langle x_{i+1}, x_i \rangle_p \leq \langle x_0, x_m \rangle_p + 2l + 8\delta$.

Quasi-convex subsets. Let Y be a subset of X . Let $\alpha \geq 0$. We denote by $Y^{+\alpha}$, the α -neighborhood of Y , i.e. the set of points $x \in X$ such that $d(x, Y) \leq \alpha$. Let $\eta \geq 0$. A point p of Y is an η -projection of $x \in X$ on Y if $|x - p| \leq d(x, Y) + \eta$. A 0-projection is simply called a *projection*.

Definition 2.5. Let $\alpha \geq 0$. A subset Y of X is α -quasi-convex if for every $x \in X$ and for every $y, y' \in Y$ the inequality $d(x, Y) \leq \langle y, y' \rangle_x + \alpha$ holds.

Lemma 2.6 (Projection on a quasi-convex). [9, Chapitre 10, Proposition 2.1] *Let Y be an α -quasi-convex subset of X . Let $x, x' \in X$.*

- (i) *If p is an η -projection of x on Y , then for all $y \in Y$, $\langle x, y \rangle_p \leq \alpha + \eta$.*

(ii) If p and p' are respective η - and η' -projections of x and x' on Y , then

$$|p - p'| \leq \max \left\{ |x - x'| - |x - p| - |x' - p'| + 2\varepsilon, \varepsilon \right\},$$

where $\varepsilon = 2\alpha + \eta + \eta' + \delta$.

Lemma 2.7 (Neighborhood of a quasi-convex). [9, Chapitre 10, Proposition 1.2] *Let $\alpha \geq 0$. Let Y be an α -quasi-convex subset of X . For every $A \geq \alpha$, the A -neighborhood of Y is 2δ -quasi-convex.*

Definition 2.8. Let Y be a subset of X . The *hull* of Y denoted by $\text{hull}(Y)$ is the union of all $(1, \delta)$ -quasi-geodesics joining two points of Y .

Lemma 2.9. [10, Lemma 2.18] *The hull of any subset of X is 6δ -quasi-convex.*

Lemma 2.10. [10, Lemma 2.19] *Let Y and Z be two subsets of X . Let x be a point of X . Assume that for all $y \in Y$ and for all $z \in Z$, the inequality $\langle y, z \rangle_x \leq \alpha$ holds. Then for all $y \in \text{hull}(Y)$ and for all $z \in \text{hull}(Z)$, we have that $\langle y, z \rangle_x \leq \alpha + 3\delta$.*

Isometries of a hyperbolic space. Let x be a point of X . An isometry g of X is either

- ▶ *elliptic*, i.e. the orbit $\langle g \rangle x$ is bounded,
- ▶ *loxodromic*, i.e. the map from \mathbf{Z} to X that sends m to $g^m x$ is a quasi-isometric embedding,
- ▶ or *parabolic*, i.e. it is neither loxodromic or elliptic.

Note that these definitions do not depend on the point x . In order to measure the action of g on X , we use two translation lengths. By the *translation length* $[g]_X$ (or simply $[g]$) we mean

$$[g]_X := \inf_{x \in X} |gx - x|.$$

The *asymptotic translation length* $[g]_X^\infty$ (or simply $[g]^\infty$) is

$$[g]_X^\infty := \lim_{n \rightarrow +\infty} \frac{1}{n} |g^n x - x|.$$

These two lengths are related as follows.

Proposition 2.11. [9, Chapitre 10, Proposition 6.4] *Let g be an isometry of X . Its translation lengths satisfy*

$$[g]^\infty \leq [g] \leq [g]^\infty + 16\delta.$$

The isometry g is loxodromic if and only if its asymptotic translation length is positive [9, Chapitre 10, Proposition 6.3]. In this case g fixes exactly two points of ∂X [9, Chapitre 10, Proposition 6.6] which are

$$g^- := \lim_{n \rightarrow -\infty} g^n x \text{ and } g^+ := \lim_{n \rightarrow +\infty} g^n x.$$

Recall that L_S is the parameter given by the stability of quasi-geodesics (see Definition 2.3).

Definition 2.12. Let g be a loxodromic isometry of X . We denote by Γ_g the union of all $L_S \delta$ -local $(1, \delta)$ -quasi-geodesics joining g^- to g^+ . The *cylinder* of g , denoted by Y_g , is the 20δ -neighborhood of Γ_g .

Remark. Note that if g is a loxodromic isometry of X , then both Γ_g and Y_g are invariant under the $\langle g \rangle$ -action.

Lemma 2.13. [10, Lemma 2.31] *Let g be a loxodromic isometry of X . The cylinder of g is 2δ -quasi-convex.*

Lemma 2.14. *Let g be a loxodromic isometry of X . For every $x \in X$, $|gx - x| \leq [g] + 2d(x, Y_g) + 112\delta$.*

Proof. Let us denote by A_g the set of points $z \in X$ such that $|gz - z| < [g] + 8\delta$. It is known that Y_g lies in the 52δ -neighborhood of A_g [10, Lemma 2.32], implying that for $y \in Y_g$, $|y - gy| \leq [g] + 112\delta$. Let x be a point of X and y a η -projection of x on Y_g . It follows that

$$\begin{aligned} |gx - x| &\leq |gx - gy| + |gy - y| + |y - x| \\ &\leq d(gx, Y_g) + \eta + |gy - y| + d(x, Y_g) + \eta \leq [g] + 112\delta + 2d(x, Y_g) + 2\eta. \end{aligned}$$

The last inequality holds for every $\eta > 0$ which completes the proof. \square

Definition 2.15. Let g be an isometry of X . Let $l \geq 0$. A path $\gamma: \mathbf{R} \rightarrow X$ is called an l -nerve of g if there exists $T \in \mathbf{R}$ with $[g] \leq T \leq [g] + l$ such that γ is a T -local $(1, l)$ -quasi-geodesic and for every $t \in \mathbf{R}$, $\gamma(t + T) = g\gamma(t)$. The parameter T is called the *fundamental length* of γ .

Remark. For every $l > 0$, one can construct an l -nerve of g as follows. Let $\eta > 0$. There exists $x \in X$ such that $|gx - x| < [g] + \eta$. Let $\gamma: [0, T] \rightarrow X$ be a $(1, \eta)$ -quasi-geodesic joining x to gx . In particular $[g] \leq T < [g] + 2\eta$. We extend γ into a path $\gamma: \mathbf{R} \rightarrow X$ in the following way: for every $t \in [0, T)$, for every $m \in \mathbf{Z}$, $\gamma(t + mT) = g^m \gamma(t)$. It turns out that γ is a T -local $(1, 2\eta)$ -quasi-geodesic. Thus if η is chosen sufficiently small then γ is an l -nerve.

This kind of path will be used to simplify some proofs. Let γ be a δ -nerve of g . If $[g] > L_S\delta$ (in particular g is loxodromic) then γ is contained in $\Gamma_g \subseteq Y_g$. By stability of quasi-geodesics (Corollary 2.2 (iii)) γ is actually 9δ -quasi-convex. Moreover it joins g^- to g^+ . By Corollary 2.2 (iv), any other $(1, \delta)$ -quasi-geodesic γ' joining g^- and g^+ is at Hausdorff distance at most 7δ of γ . Thus Y_g lies in the 27δ -neighborhood of γ . Hence it provides a g -invariant line than can advantageously be used as a substitution for a cylinder.

3 Rotation families

Original settings. In this section we extend the framework of rotation families given by F. Dahmani, V. Guirardel and D. Osin in [11]. Let G be a group acting by isometries on a δ -hyperbolic length space X .

Definition 3.1. Let $\sigma > 0$. A σ -rotation family is a non-empty collection \mathcal{R} of pairs (H, v) where H is a subgroup of G and v a point of X satisfying the following properties.

- (R1) For every $(H, v) \in \mathcal{R}$, for every $x \in B(v, \sigma/10)$, and for every $h \in H \setminus \{1\}$, the equality $|hx - x| = 2|v - x|$ holds.
- (R2) For every $(H, v), (H', v') \in \mathcal{R}$, if $(H, v) \neq (H', v')$, then $|v - v'| \geq \sigma$.
- (R3) For all $g \in G$ and for all $(H, v) \in \mathcal{R}$, the pair (gHg^{-1}, gv) belongs to \mathcal{R} . In particular, \mathcal{R} has a natural structure of G -set.

Remark. It follows from (R2) and (R3) that for every $(H, v) \in \mathcal{R}$, H is actually a normal subgroup of $\text{Stab}(v)$.

Notations. Let $(H, v) \in \mathcal{R}$. The idea is that each element $h \in H$ acts on X like a rotation of center v and very large angle - see Axiom (R1). Therefore v is called an *apex* and H a *rotation group*. If \mathcal{S} is a subset of \mathcal{R} denote by $v(\mathcal{S})$ the set of all apices v such that $(H, v) \in \mathcal{S}$. Similarly $H(\mathcal{S})$ stands for the set of all rotation groups H with $(H, v) \in \mathcal{S}$. Given a subset Y of X , we denote by K_Y the subgroup of G generated by all the rotation groups H where $(H, v) \in \mathcal{R}$ and $v \in Y$. The (normal) subgroup generated by all the rotation groups is simply denoted by K .

In their work [11], F. Dahmani, V. Guirardel and D. Osin use the properties of such a family to study the structure of K and the quotient $\bar{G} = G/K$. Among other things, they prove the following facts. See also [10] for a slightly different exposition of the last two points.

Theorem 3.2. *There exists $\sigma_0 > 0$ which only depends on δ , such that for every $\sigma \geq \sigma_0$, and every σ -rotating family \mathcal{R} , the following holds.*

- (i) *There exists a subset \mathcal{S} of \mathcal{R} such that K is isomorphic to the free product of the element of $H(\mathcal{S})$.*
- (ii) *The subgroup K acts properly on $X \setminus v(\mathcal{R})$.*
- (iii) *The quotient $\bar{X} = X/K$ is a $\bar{\delta}$ -hyperbolic length space with $\bar{\delta} \leq 900\delta$.*

For a proof of the Theorem 3.2, see [11, Theorem 5.3] for (i) (or Theorem 3.3 below), [10, Corollary 3.12] for (ii) and [10, Propositions 3.14 and 3.18] for (iii) (or, [11, Proposition 5.28] for (ii) and (iii)).

Extended windmill. The goal of this section is to improve the approach of F. Dahmani, V. Guirardel and D. Osin in order to study the structure of the subgroup of G generated by K and some subgroup of G . More precisely we prove the following statement.

Theorem 3.3. *There exists $\sigma_0 > 0$ which only depends on δ such that the following holds. Assume that $\sigma \geq \sigma_0$. Let \mathcal{R} be an α -rotating family. Let Y be a 2δ -quasi-convex subset of X and N a subgroup of G stabilizing Y with the following properties*

- (i) *For every $(H, v) \in \mathcal{R}$, for every $h \in H \setminus \{1\}$, for every $y, y' \in Y$, $\langle y, hy' \rangle_v \leq 100\delta$.*
- (ii) *For every $(H, v) \in \mathcal{R}$, $\text{Stab}(v) \cap N = \{1\}$.*

Then there exists a subset \mathcal{S} of \mathcal{R} such that the subgroup generated by N and K is isomorphic to the free product of N and the elements of $H(\mathcal{S})$.

If N is trivial then we recover the first point of Theorem 3.2. The rest of this section is dedicated to the proof of the theorem. For that, we extend the notion of windmill (see Definition 3.6). But first we need to define σ_0 . Applying to Proposition 2.4 with $l = 105\delta$ there exists $\sigma_0 = L(105\delta, \delta)$ such that for any sequence of points y_0, \dots, y_{m+1} in X , satisfying that

- (i) for every $i \in \{1, \dots, m\}$, $\langle y_{i+1}, y_{i-1} \rangle_{y_i} \leq 105\delta$,

(ii) for every $i \in \{1, \dots, m-1\}$, $|y_{i+1} - y_i| \geq \sigma_0$,

then, the inequality $\langle y_0, y_{m+1} \rangle_{y_i} \leq 110\delta$ holds for every $i \in \{0, \dots, m+1\}$. Moreover, for all $x \in X$, there exists $i \in \{0, \dots, m\}$, such that $\langle y_{i+1}, y_i \rangle_x \leq \langle y_0, y_{m+1} \rangle_x + 218\delta$. Without loss of generality we can assume that σ_0 is greater than $10^{10}\delta$.

From now on we assume that \mathcal{R} is a σ -rotation family with $\sigma \geq \sigma_0$. Let us recall now some basic facts.

Lemma 3.4. [10, Lemma 3.3] *Let $(H, v) \in \mathcal{R}$. Let $h \in H \setminus \{1\}$. For every $x \in X$, $\langle x, hx \rangle_v \leq 2\delta$.*

Lemma 3.5. *Let $(H, v) \in \mathcal{R}$. Let $h \in H \setminus \{1\}$. Let Y be an α -quasi-convex subset of X such that $d(v, Y) > \alpha + 3\delta$. For every $y, y' \in Y$ we have $\langle y, hy' \rangle_v \leq 3\delta$.*

Proof. Let $y, y' \in Y$. Combining the four point inequality (1) with Lemma 3.4 we get

$$\min \{ \langle y, y' \rangle_v, \langle y, hy' \rangle_v \} \leq \langle y', hy' \rangle_v + \delta \leq 3\delta.$$

Recall that since Y is α -quasi-convex (Definition 2.5), we have $\langle y, y' \rangle_v \geq d(v, Y) - \alpha > 3\delta$. Consequently, the minimum cannot be achieved by $\langle y, y' \rangle_v$, and hence $\langle y, hy' \rangle_v \leq 3\delta$. \square

Definition 3.6. Let W be a subset of X , N a subgroup of G and V a subset of $v(\mathcal{R})$. Let L be the subgroup of G generated by N and K_V . The triple (W, N, V) is an *extended windmill* if the following holds.

(W1) W is 2δ -quasi-convex

(W2) W and V are L -invariant.

(W3) For every $(H, v) \in \mathcal{R}$ such that $v \notin V$, for every $h \in H \setminus \{1\}$, for every $x, x' \in W$, $\langle x, hx' \rangle_v \leq 100\delta$.

(W4) For every $(H, v) \in \mathcal{R}$ such that $v \notin V$, $\text{Stab}(v) \cap L = \{1\}$.

Remark. If N is the trivial group and V is the set of apices contained in W , then we roughly recover the definition of windmill given in [11]. For our purpose, V may be a smaller set. However, if an apex v is not contained in V then the corresponding rotation group H should rotate the points of W with a “large angle”.

Lemma 3.7. *If (W, N, V) is an extended windmill, then V is contained in the 4δ -neighbourhood of W .*

Proof. Let $(H, v) \in \mathcal{R}$, $v \in V$, and $h \in H \setminus \{1\}$. Let $y \in W$. By Lemma 3.4, $\langle y, hy \rangle_v \leq 2\delta$. By Axiom (W2), W is K_V -invariant and it follows $hy \in W$. By Axiom (W1), W is 2δ -quasi-convex subset of X , and therefore, $d(v, W) \leq \langle y, hy \rangle_v + 2\delta \leq 4\delta$. \square

Proposition 3.8. *Let (W, N, V) be an extended windmill. Let L be the subgroup generated by N and K_V . There exists a subset W' of X with the following properties.*

(a) *The $(\sigma/10)$ -neighbourhood of W is contained in W' .*

(b) *The triple (W', N, V') is an extended windmill, where $V' = W' \cap v(\mathcal{R})$.*

(c) *There exists a subset \mathcal{R}_0 of \mathcal{R} such that the subgroup L' generated by N and $K_{V'}$ is isomorphic to the following free product*

$$L' = L * (*_{(H,v) \in \mathcal{R}_0} H)$$

Proof. Let us denote by A the following set of apices

$$A = \left\{ v \in v(\mathcal{R}) \setminus V \mid d(v, W) \leq 3\sigma/10 \right\}.$$

We consider two cases, depending on whether A is empty or not.

Case 1. *Assume that A empty.* We choose for W' the $\sigma/10$ -neighborhood of W . Clearly (a) holds. By Lemma 3.7 and since σ is much greater than δ , we get that $V' = W' \cap v(\mathcal{R}) = V$. To see that (W', N, V) is an extended windmill, observe that (W1) follows by Lemma 2.7; (W2) and (W4) follows trivially since $V = V'$ then $L' = \langle N, K_{V'} \rangle = L$ and (W, N, V) is an extended windmill. It remains (W3), which follows from Lemma 3.5 and bearing in mind that $A = \emptyset$. Note also that (c) holds because $L = L'$.

Case 2. *Assume that A is not empty.* We denote by S (like *sail*) the hull of $W \cup A$ (see Definition 2.8). We are going to let this sail “turn” around the apices of A . Let W' be the $\sigma/10$ -neighborhood of $K_A \cdot S$ and $V' = W' \cap v(\mathcal{R})$. In particular, W' contains the $\sigma/10$ -neighborhood of W , and hence (a) holds. The goal is to prove that (b) and (c) hold. The following observation will be useful: since V and W are both L -invariant, hence so is A (and thus S) and thus

$$W' \text{ is } \langle L, K_A \rangle\text{-invariant, } L \text{ normalizes } K_A \text{ and hence } \langle L, K_A \rangle \cdot S = K_A \cdot S. \quad (4)$$

Lemma 3.9. *Let $(H, v) \in \mathcal{R}$ such that $v \in A$. Let $x, y \in S$ and $h \in H \setminus \{1\}$. Then $\langle x, hy \rangle_v \leq 105\delta$.*

Proof. Recall that S is the hull of $W \cup A$. According to Lemma 2.10, it is sufficient to prove that for all $x, y \in W \cup A$, $\langle x, hy \rangle_v \leq 102\delta$. Let $x, y \in W \cup A$. Note that if $x = v$ or $y = v$, then $\langle x, hy \rangle_v = 0$ (h fixes v). Therefore we can suppose that x and y are distinct from v , and we have 3 different cases.

Case 1. *Assume that x and y lie in W .* Recall that v does not belong to V , thus by Axiom (W3), $\langle x, hy \rangle_v \leq 100\delta$.

Case 2. *Assume that x lies in W and y in $A - \{v\}$.* We denote by r and q δ -projections of v and y on W respectively. We claim that $\langle y, q \rangle_v > 101\delta$. By Lemma 2.6 (ii),

$$|q - r| \leq \max \{ |y - v| - |y - q| - |v - r| + 14\delta, 7\delta \}.$$

However y and v are two distinct apices. It follows that $|y - v| \geq \sigma$ whereas $|y - q|$ and $|v - r|$ are at most $3\sigma/10 + \delta$. The triangle inequality combined with our choice of σ_0 yields $|q - r| > 7\delta$. Consequently we necessarily have

$$|y - q| + |q - v| \leq |y - q| + |q - r| + |r - v| \leq |y - v| + 14\delta.$$

In particular, $\langle y, v \rangle_q \leq 7\delta$. Hence $\langle y, q \rangle_v = |q - v| - \langle y, v \rangle_q \geq |q - v| - 7\delta$. By the triangle inequality, $|q - v| \geq |y - v| - |y - q|$. Since $|y - v| \geq \sigma$ and $|y - q| \leq 3\sigma/10 + \delta$, our claim follows from $\sigma \geq \sigma_0$.

Recall that $x, q \in W$. It follows from Axiom (W3) that $\langle x, hq \rangle_v \leq 100\delta$. The four point inequality leads to

$$\min \{ \langle x, hy \rangle_v, \langle hy, hq \rangle_v \} \leq \langle x, hq \rangle_v + \delta \leq 101\delta. \quad (5)$$

Note that $\langle y, q \rangle_v = \langle hy, hq \rangle_v$. Equation (5) combined with our claim gives that $\langle x, hy \rangle_v \leq 101\delta$.

Case 3. Assume that x and y lie in $A - \{v\}$. Again, denote by q a δ -projection of y on W . Since the lemma holds in Case 2, we get that $\langle x, hq \rangle_v \leq 101\delta$. Using again the four point inequality, (5), and the previous argument about $\langle hy, hq \rangle_v$, we get that $\langle x, hy \rangle_v \leq 102\delta$. \square

Lemma 3.10. *Let $v \in v(\mathcal{R})$. If $d(v, S) \leq \sigma/5$ then $v \in V \cup A$.*

Proof. Let p be a δ -projection of v on S . By the definition of hull, there exists $y, y' \in W \cup A$ such that p lies on a $(1, \delta)$ -quasi-geodesic γ with endpoint y, y' . In particular, the triangle inequality yields to $\langle y, y' \rangle_v \leq \langle y, y' \rangle_p + |p - v|$, and hence $\langle y, y' \rangle_v \leq \sigma/5 + 2\delta$. Let us denote by z and z' respective δ -projections of y and y' on W . Applying twice the four point inequality (1) we get

$$\min \{ \langle y, z \rangle_v, \langle z, z' \rangle_v, \langle y', z' \rangle_v \} \leq \langle y, y' \rangle_v + 2\delta \leq \sigma/5 + 4\delta. \quad (6)$$

Assume first that the minimum in (6) is achieved by $\langle z, z' \rangle_v$. The windmill W being 2δ -quasi-convex, we have

$$d(v, W) \leq \langle z, z' \rangle_v + 2\delta \leq \sigma/5 + 6\delta \leq 3\sigma/10.$$

By definition of A , v is necessarily a point of $V \cup A$.

Assume now that the minimum is achieved by $\langle y, z \rangle_v$ (the proof works similarly for $\langle y', z' \rangle_v$). It follows from the triangle inequality that $|y - v| - |y - z| \leq \langle y, z \rangle_v \leq \sigma/5 + 4\delta$. If $y \in W$, $|y - z| \leq \delta$ and $|y - v| \leq \sigma/5 + 5\delta$ and by definition of A , $v \in V \cup A$. If $y \in A$, by construction $|y - z|$ is bounded above by $3\sigma/10 + \delta$, thus $|y - v| < \sigma$. Since the distance between two distinct apices of \mathcal{R} is at least σ , we get that $y = v$. Hence $v \in A$. \square

Lemma 3.11. *The sets $L \cup K_A$ and $N \cup K_{V'}$ generate the same subgroup L' of G . Moreover W' and V' are L' -invariant.*

Remark. This lemma proves Axiom (W2) for our new windmill.

Proof. By construction $A \subseteq V'$. Since $L = \langle N, K_V \rangle$, we have that $\langle L \cup K_A \rangle \subset \langle N, K_{V'} \rangle$. It is enough to show that $K_{V'} \subset \langle L \cup K_A \rangle$. It follows from Lemma 3.10 that every apex contained in the $\sigma/10$ -neighborhood of $K_A \cdot S$ (i.e. W') actually belongs to $K_A \cdot S$. Thus V' is the set $K_A \cdot (V \cup A)$. Recall that \mathcal{R} is G -invariant. The other inclusion follows. Moreover by (4), W' and V' are both L' -invariant. \square

For the remainder of the section, L' denotes the subgroup $\langle L, K_A \rangle = \langle N, K_{V'} \rangle$.

Decomposition of the elements of L' . We denote by \mathbf{A} a set of representatives for A/L . We use \mathbf{L} to denote an abstract copy of L , and similarly for $(H, v) \in \mathcal{R}$, we use (\mathbf{H}, \mathbf{v}) to denote an abstract copy of the pair. We denote by \mathbf{L}' the free product of \mathbf{L} and the rotation groups \mathbf{H} where $(\mathbf{H}, \mathbf{v}) \in \mathcal{R}$ and $\mathbf{v} \in \mathbf{A}$.

$$\mathbf{L}' = \mathbf{L} * (*_{(\mathbf{H}, \mathbf{v}) \in \mathcal{R}, \mathbf{v} \in \mathbf{A}} \mathbf{H}).$$

It comes with a natural morphism $\mathbf{L}' \rightarrow L'$. By construction this map is onto. We are going to prove (among other things) that it is an isomorphism. Let \mathbf{g} be an element of \mathbf{L}' . It can be written $g = \mathbf{u}_0 \mathbf{h}_1 \mathbf{u}_1 \dots \mathbf{u}_{m-1} \mathbf{h}_m \mathbf{u}_m$, where

- (i) for every $i \in \{1, \dots, m\}$ there exists $(\mathbf{H}_i, \mathbf{v}_i) \in \mathcal{R}$ with $\mathbf{v}_i \in \mathbf{A}$ such that $\mathbf{h}_i \in \mathbf{H}_i \setminus \{1\}$,
- (ii) for every $i \in \{0, \dots, m\}$, $\mathbf{u}_i \in \mathbf{L}$,
- (iii) for every $i \in \{1, \dots, m-1\}$, if $\mathbf{u}_i = 1$ then $\mathbf{v}_i \neq \mathbf{v}_{i+1}$.

The integer m does not depend on the decomposition above. We call it the *number of rotation of \mathbf{g}* and denote it by $m(\mathbf{g})$. The image g of \mathbf{g} in L' can be rewritten as follows

$$\begin{aligned} g &= [\mathbf{u}_0 \mathbf{h}_1 \mathbf{u}_0^{-1}] [(\mathbf{u}_0 \mathbf{u}_1) \mathbf{h}_2 (\mathbf{u}_0 \mathbf{u}_1)^{-1}] \dots [(\mathbf{u}_0 \dots \mathbf{u}_{m-1}) \mathbf{h}_m (\mathbf{u}_0 \dots \mathbf{u}_{m-1})^{-1}] \mathbf{u}_0 \dots \mathbf{u}_m \\ &= h_1 \dots h_m u, \end{aligned}$$

where $u = \mathbf{u}_0 \dots \mathbf{u}_m$ is in L and for every $i \in \{1, \dots, m\}$, $h_i = (\mathbf{u}_0 \dots \mathbf{u}_{i-1}) \mathbf{h}_i (\mathbf{u}_0 \dots \mathbf{u}_{i-1})^{-1}$ is an element of the rotation group H_i fixing the vertex $v_i = \mathbf{u}_0 \dots \mathbf{u}_{i-1} \mathbf{v}_i$. Since A is L -invariant, all the apices v_i belongs to A . We claim that for every $i \in \{1, \dots, m-1\}$, $v_i \neq v_{i+1}$. Let $i \in \{1, \dots, m-1\}$. Assume that on the contrary our claim is false. Then $\mathbf{v}_i = \mathbf{u}_i \mathbf{v}_{i+1}$. The points \mathbf{v}_i and \mathbf{v}_{i+1} both belongs to \mathbf{A} , the set of representatives of A/L . It follows that $\mathbf{v}_i = \mathbf{v}_{i+1}$ is fixed by \mathbf{u}_i . According to Axiom (W4), \mathbf{u}_i is necessarily trivial. It contradicts property (iii) of the decomposition of \mathbf{g} in \mathbf{L}' .

The second way of writing the elements of L' , namely $g = h_1 h_2 \dots h_m u$, is shorter and will be preferred and used in the next lemma. Note that the integer m that appears in the second form is still the number of rotations of \mathbf{g} .

Lemma 3.12. *Let $y, y' \in S$. Let $\mathbf{g} \in \mathbf{L}'$, g its image in L' and m its number of rotations. There exists a sequence of points $y = y_0, \dots, y_{m+1} = gy'$ of X satisfying the following properties*

- (i) for all $i \in \{1, \dots, m+1\}$ there exists $g_i \in L'$ such that $g_i^{-1} y_{i-1}$ and $g_i^{-1} y_i$ belong to S ,
- (ii) for all $i \in \{1, \dots, m-1\}$, $|y_{i+1} - y_i| \geq \sigma$,
- (iii) for all $i, j, k \in \{1, \dots, m\}$ with $i \leq j \leq k$ we have $\langle y_i, y_k \rangle_{y_j} \leq 110\delta$,
- (iv) For all $x \in X$ there exists $i \in \{0, \dots, m\}$ such that $\langle y_{i+1}, y_i \rangle_x \leq \langle y, gy' \rangle_x + 218\delta$.

Proof. According to our previous discussion g can be written $h_1 \dots h_m u$ where $u \in L$ and for every $i \in \{1, \dots, m\}$ there exists $(H_i, v_i) \in \mathcal{R}$ with $v_i \in A$ such that $h_i \in H_i \setminus \{1\}$. Moreover two consecutive apices v_i and v_{i+1} are distinct. If $m = 0$, i.e. g belongs to L , then the points $y_0 = y$ and $y_1 = gy'$ lie in S and hence satisfy the conclusion of the lemma. Assume now that $m \geq 1$. For all $i \in \{1, \dots, m\}$, we put $g_i = h_1 \dots h_{i-1}$ and $y_i = g_i v_i$. Moreover, we put $g_{m+1} = h_1 \dots h_m = gu^{-1}$, $y_0 = y$ and $y_{m+1} = gy'$.

- Let $i \in \{1, \dots, m+1\}$. If $i \neq m+1$ then $g_i^{-1}y_i = v_i$. Otherwise $g_{m+1}^{-1}y_{m+1} = uy'$. However u belongs to L which stabilizes S . Thus $g_i^{-1}y_i$ belongs to S . Assume now that $i \neq 1$. Recall that h_{i-1} fixes v_{i-1} thus $g_i^{-1}y_{i-1} = h_{i-1}^{-1}v_{i-1} = v_{i-1}$. On the other hand $g_1^{-1}y_0 = y$. By construction $g_i^{-1}y_{i-1}$ is a point of S . This completes the proof of Point (i).
- Let $i \in \{1, \dots, m-1\}$. The apex v_i is fixed by h_i therefore

$$|y_{i+1} - y_i| = |g_i h_i v_{i+1} - g_i v_i| = |v_{i+1} - v_i|.$$

However v_{i+1} and v_i are two distinct apices of \mathcal{R} , therefore $|y_{i+1} - y_i| \geq \sigma$. This proves Point (ii).

- Let $i \in \{1, \dots, m\}$. By construction $g_i^{-1}y_i = v_i$ whereas $g_i^{-1}y_{i-1}$ belongs to S . On the other hand $g_i^{-1}y_{i+1} = h_i g_{i+1}^{-1}y_{i+1}$. By Lemma 3.9, $\langle y_{i-1}, y_{i+1} \rangle_{y_i} = \langle g_i^{-1}y_{i-1}, g_i^{-1}y_{i+1} \rangle_{v_i}$ is bounded above by 105δ .

We chose the constant σ big enough compared to δ in such a way that we can apply Proposition 2.4 to the sequence y_0, \dots, y_{m+1} . Point (iii) and Point (iv) follow from the stability of discrete quasi-geodesics. \square

Lemma 3.13. *The set $K_A \cdot S$ is 224δ -quasi-convex whereas W' is 2δ -quasi-convex.*

Remark. This lemma proves Axiom (W1) for our new windmill. Let v be an apex of $v(\mathcal{R})$ which is not in V' . According to Lemma 3.10 we have $d(v, W') \geq \sigma/10$. Since W' is quasi-convex, it follows from Lemma 3.5 that our new windmill satisfies Axiom (W3).

Proof. The set W' was defined as the $\sigma/10$ -neighborhood of $K_A \cdot S$. According to Lemma 2.7, it is sufficient to show that $K_A \cdot S$ is 224δ -quasi-convex. Let $x \in X$ and $y, y' \in K_A \cdot S$. It follows from Lemma 3.12 that there exist $z, z' \in S$ and $g \in L'$ such that $\langle gz, gz' \rangle_x \leq \langle y, y' \rangle_x + 218\delta$. However S being a hull, it is 6δ -quasi-convex (Lemma 2.9). In particular so is gS . By (4), we have that $gS \subseteq K_A \cdot S$. Therefore

$$d(x, K_A \cdot S) \leq d(x, gS) \leq \langle gz, gz' \rangle_x + 6\delta \leq \langle y, y' \rangle_x + 224\delta. \quad \square$$

Lemma 3.14. *Let \mathbf{g} be an element of \mathbf{L}' and g its image in L' . One of the following holds*

- (i) \mathbf{g} belongs to \mathbf{L} (and thus $g \in L$).
- (ii) There exists $(H, v) \in \mathcal{R}$, with $v \in A$ such that $g \in H \setminus \{1\}$.
- (iii) For every $y \in S$, $|gy - y| \geq \sigma - 440\delta$.

Proof. Let m be the rotation number of \mathbf{g} . Suppose that $m = 0$. Then \mathbf{g} belongs to \mathbf{L} , which gives the first case. Suppose that $m = 1$. There exists $(H, v) \in \mathcal{R}$ with $v \in A$, $h \in H \setminus \{1\}$ and $u \in L$ such that $g = hu$. If $u = 1$, then g belongs to $H \setminus \{1\}$, which gives the second case. Therefore we can assume that $u \neq 1$. Since u belongs to L , Axiom (W4) yields $uv \neq v$. Let $y \in S$. It follows from the triangle inequality that

$$|v - y| + |uy - v| = |uv - uy| + |uy - v| \geq |uv - v| \geq \sigma$$

On the other hand, both y and uy belong to S . By [Lemma 3.9](#), we get $\langle huy, y \rangle_v \leq 105\delta$. Hence,

$$|gy - y| = |huy - y| \geq |huy - v| + |v - y| - 210\delta = |uy - v| + |v - y| - 210\delta \geq \sigma - 210\delta,$$

which gives the third case. We have proved the lemma when $m \leq 1$.

Suppose now that $m \geq 2$. Let $y \in S$. According to [Lemma 3.12](#), there exists a sequence of points $y = y_0, \dots, y_{m+1} = gy$ with the following properties.

- ▶ $\langle y_0, y_{m+1} \rangle_{y_1} \leq 110\delta$ and $\langle y_1, y_{m+1} \rangle_{y_2} \leq 110\delta$.
- ▶ $|y_1 - y_2| \geq \sigma$.

In particular,

$$|gy - y| \geq |y_0 - y_1| + |y_1 - y_2| + |y_2 - y_{m+1}| - 440\delta \geq \sigma - 440\delta. \quad \square$$

Lemma 3.15. *For every $(H, v) \in \mathcal{R}$ with $v \notin V'$, we have $\text{Stab}(v) \cap L' = \{1\}$.*

Remark. The conclusion of the lemma corresponds to Axiom [\(W4\)](#) for our new windmill. Recall that Axiom [\(W1\)](#), [\(W2\)](#) and [\(W3\)](#) have been already proved. It finishes the proof that the statement (b) of the proposition holds.

Proof. Let $(H, v) \in \mathcal{R}$ such that $v \notin V'$. Let $g \in L'$ such that $gv = v$. Let y be a δ -projection of v on $K_A \cdot S$. There exists $w \in K_A$ such that wy belongs to S . Recall that V' is K_A -invariant. Note that by [Lemma 3.7](#) and [Lemma 3.10](#), we get that V is a subset of V' . Thus wgw^{-1} fixes the apex wv which does not belong to V' and neither to V . We apply [Lemma 3.14](#) to wgw^{-1} . We distinguish three cases.

Assume first that $wgw^{-1} \in L$. Since wgw^{-1} fixes an apex $wv \notin V$, Axiom [\(W4\)](#), implies that g is trivial.

Assume now that there exists $(H', v') \in \mathcal{R}$ with $v' \in A$ such that $wgw^{-1} \in H' \setminus \{1\}$. A non trivial element of a rotation groups fixes exactly one points. However wgw^{-1} fixes $v' \in V'$ and $wv \notin V'$. This case never happens.

The last case states that $|gy - y| = |(wgw^{-1})wy - wy| \geq \sigma - 440\delta$. The points y and gy are respective δ -projections of v and gv on $K_A \cdot S$, which is 224δ -quasi-convex. It follows from [Lemma 2.6](#) that

$$|gy - y| \leq \max \{|gv - v| - 2|v - y| + 902\delta, 451\delta\}.$$

Since $|gy - y| \geq \sigma - 440\delta$ we get $|gv - v| \geq \sigma - 1342\delta$. Thus g cannot fix v . This case also never happens. \square

Lemma 3.16. *The canonical map $\mathbf{L}' \rightarrow L'$ is one-to-one.*

Proof. Let \mathbf{g} be an element of \mathbf{L}' whose image g in L' is trivial. It follows from [Lemma 3.14](#) that \mathbf{g} belongs to \mathbf{L} . By construction the map $\mathbf{L}' \rightarrow L'$ induces an embedding of \mathbf{L} into L' , hence $\mathbf{g} = 1$. \square

Remark. Lemma 3.16 shows that property (c) holds and concludes the proof of Proposition 3.8. \square

Proof of Theorem 3.3. Let \mathcal{R} be σ -rotation family, and let Y and N as in the hypothesis of the theorem. Note that (Y, N, \emptyset) is an extended windmill, which we denote by (W_0, N, V_0) . A proof by induction using Proposition 3.8 shows that for every $n \in \mathbf{N}$ there is an extended windmill (W_n, N, V_n) with the following property. If L_n stands for the subgroup generated by N and K_{V_n} , then for every $n \in \mathbf{N} \setminus \{0\}$,

- (i) W_n contains the $\sigma/10$ -neighborhood of W_{n-1} ;
- (ii) $V_n = W_n \cap v(\mathcal{R})$;
- (iii) there exists a subset \mathcal{R}_n of \mathcal{R} such that L_n is isomorphic to the free product of L_{n-1} and the rotation groups of $H(\mathcal{R}_n)$.

Note also that $L_0 = N$. Since the sequence of subsets (W_n) is growing every vertex of $v(\mathcal{R})$ ultimately belongs to some V_n . In other words (L_n) is an increasing sequence of subgroups of G whose union L is exactly the subgroup generated by N and K . Let \mathcal{S} be the union of all \mathcal{R}_n . It follows from the free product structure of every L_n that L is isomorphic to the free product of N and the rotation groups of $H(\mathcal{S})$. \square

Application. The goal of this paragraph is to prove the following statement.

Theorem 3.17. *Let X be δ -hyperbolic space and G a group acting by isometries on it. There exists $\sigma_0 > 0$ with the following property. Let \mathcal{R} be a σ -rotation family with $\sigma > \sigma_0$. Let K be the (normal) subgroup generated by all the rotation groups of $H(\mathcal{R})$ and \bar{G} be the quotient G/K . Then the following holds.*

- (i) *The quotient $\bar{X} = X/K$ is $\bar{\delta}$ -hyperbolic with $\bar{\delta} \leq 900\delta$.*
- (ii) *For every $\bar{g} \in \bar{G}$ acting loxodromically on \bar{X} , there exists a pre-image $g \in G$ of \bar{g} and a subset \mathcal{S} of \mathcal{R} such that*

$$\langle g, K \rangle = \langle g \rangle * (*_{H \in H(\mathcal{S})} H).$$

Let $\delta > 0$. From now on σ_0 is the maximum of the constants respectively given by Theorem 3.2 and Theorem 3.3. Up to increasing the value of σ_0 we can always assume that $\sigma_0 \geq L_S \delta + 150\delta$, where L_S is the constant of Definition 2.3. Let X be a δ -hyperbolic space endowed with an action by isometries of a group G . Let \mathcal{R} be σ -rotation family with $\sigma > \sigma_0$. Recall that for $g \in G$ that is a loxodromic isometry of G , Y_g denotes the cylinder of g (see Definition 2.12).

Definition 3.18. Let g be a loxodromic element of G . We say that g is \mathcal{R} -reduced if for every $(H, v) \in \mathcal{R}$, for every $h \in H \setminus \{1\}$, for every $y, y' \in Y_g$, $\langle hy, y' \rangle_v \leq 100\delta$.

Lemma 3.19. *Let $g \in G$. There exists $u \in K$ such that ug is either not loxodromic or \mathcal{R} -reduced.*

Proof. We assume that for every $u \in K$, ug is loxodromic. We now fix $u_0 \in K$ such that

$$\text{for every } u \in K, [u_0 g] \leq [ug] + \delta. \quad (7)$$

For simplicity we write $f = u_0 g$. The goal is to prove that f is \mathcal{R} -reduced. Assume on the contrary that it is not. There exists $(H, v) \in \mathcal{R}$, $h \in H \setminus \{1\}$ and $y, y' \in Y_f$ such that $\langle hy, y' \rangle_v > 100\delta$. We first claim that v is 5δ -close from Y_f . According to Lemma 3.4, $\langle hy, y \rangle_v \leq 2\delta$. Using the four point inequality we have

$$\min \{ \langle hy, y' \rangle_v, \langle y', y \rangle_v \} \leq \langle hy, y \rangle_v + \delta \leq 3\delta.$$

By assumption the minimum cannot be achieved by $\langle y, hy' \rangle_v$ thus $\langle y', y \rangle_v \leq 3\delta$. However Y_f is 2δ -quasi-convex, thus v is 5δ -close from Y_f . In particular $[f] \geq |fv - v| - 122\delta$ (Lemma 2.14). Since f is loxodromic, it cannot fix v . It follows from (R2) that $[f] \geq \sigma - 122\delta \geq L_S \delta$. We fix $\gamma: \mathbf{R} \rightarrow X$ a δ -nerve of f . Note that Y_f is contained in the 27δ -neighborhood of γ (see Definition 2.15 and the discussion afterwards). Let z be a point of X such that $\langle y, v \rangle_z \leq \delta$ and $|v - z| = 100\delta$ and

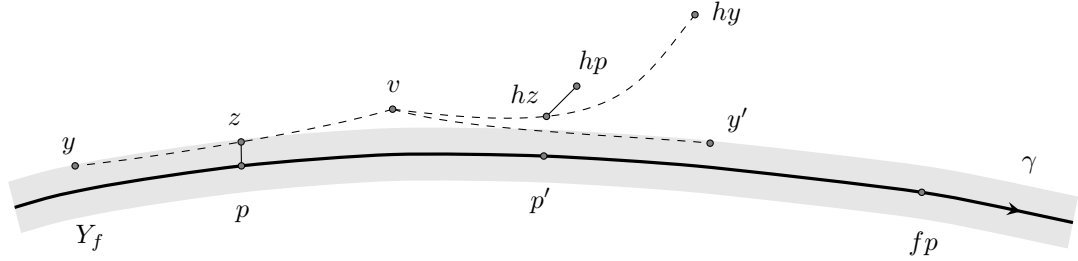


Figure 1: Shortening the translation length of f .

p a projection of z on γ (see Figure 1). Note that $|v - z| < \max\{\sigma/10, \langle hy, y' \rangle_v\}$. The points y and v both belong to the 5δ -neighborhood of Y_f which is 2δ -quasi-convex (Lemma 2.7), hence $d(z, Y_f) \leq 8\delta$. On the other hand, the cylinder Y_f is contained in the 27δ -neighborhood of γ , thus $|z - p| \leq 35\delta$.

We want to use the previous reasoning to bound $|hz - p'|$ where p' is a δ -projection of hz . For that, we need to bound $\langle y', v \rangle_{hz}$. This follows as a consequence of (3) applied to v, z, hz and y' and bearing in mind that $|v - z| < \langle y', hy \rangle_v$. Indeed, we get

$$\begin{aligned} \langle y', v \rangle_{hz} &\leq \max\{|v - hz| - \langle y', hy \rangle_v, \langle v, hy \rangle_{hz}\} + \delta \\ &= \max\{|v - z| - \langle y', hy \rangle_v, \langle v, y \rangle_z\} + \delta = 2\delta. \end{aligned}$$

Now, reasoning as previously we get $|hz - p'| \leq 36\delta$. Combined with (R1) it yields

$$|p - p'| \geq |hz - z| - 71\delta \geq 2|v - z| - 71\delta \geq 129\delta.$$

Up to changing f by its inverse we can always assume that fp and p' are in the same connected component of $\gamma \setminus \{p\}$. Recall that γ is an $[f]$ -local $(1, \delta)$ -quasi-geodesic. Thus $|fp - p'| \leq |fp - p| - |p - p'| + \delta$. However p being a point of γ , $|fp - p| \leq [f] + \delta$. On the other hand $|hp - p'| \leq |hp - hz| + |hz - p'| \leq 71\delta$. Consequently

$$[h^{-1}u_0g] = [h^{-1}f] \leq |fp - hp| \leq |fp - p'| + |hp - p'| \leq [f] - 56\delta < [u_0g] - \delta.$$

Since $h^{-1}u_0 \in K$, this last inequality contradicts (7). Hence $f = u_0g$ is \mathcal{R} -reduced. \square

Proof of Theorem 3.17. According to Theorem 3.2 the quotient space $\bar{X} = X/K$ is $\bar{\delta}$ -hyperbolic with $\bar{\delta} \leq 900\delta$. Let \bar{g} be an element of \bar{G} , loxodromic for its action on \bar{X} . In particular, $[\bar{g}]^\infty > 0$. By construction the projection $X \rightarrow \bar{X}$ is 1-Lipschitz. Thus every pre-image of \bar{g} is loxodromic. According to Lemma 3.19, there exists a preimage g of \bar{g} which is \mathcal{R} -reduced. Let Y be the cylinder of g and N the cyclic group generated by g . We need to check that the hypothesis of Theorem 3.3 hold. First, by Lemma 2.13, Y is 2δ -quasi-convex. By definition Y is N -invariant. Since g is \mathcal{R} -reduced, condition (i) of Theorem 3.3 hold. Finally, as g is loxodromic, g cannot fix a point, thus for every $(H, v) \in \mathcal{R}$, we get $\text{Stab}(v) \cap N = \{1\}$. \square

4 Relatively hyperbolic groups and rotation families

There exists several definitions for the concept of relatively hyperbolic groups, we present here to the one of Osin. We refer to [20] for details and the equivalence with the concepts of relative hyperbolicity of Bowditch and strongly relative hyperbolicity in the sense of Farb. Also see [17] for a definition of relative hyperbolicity that mimics the one of geometrically finite hyperbolic groups.

Definition 4.1. A group finitely generated group G is *hyperbolic relative to a family of subgroups* $\mathcal{P} = \{P_1, \dots, P_n\}$, if it admits a finite presentation relative to \mathcal{P} and this presentation has a linear relative Dehn function. The subgroups P_1, \dots, P_n are called *peripheral (or parabolic) subgroups* of G .

Remark. We have chosen to state the present definition for sake of conciseness. It is worth noticing that Definition 4.1 in the case that \mathcal{P} is empty, recovers the definition of an hyperbolic group in terms of isoperimetric inequalities.

The connection between relatively hyperbolicity and rotation families follows from [11, Proposition 7.7 and Corollary 7.8.], a version of which, is stated below.

Proposition 4.2. *Let G be a finitely generated group hyperbolic relative to a finite family of subgroups $\{P_1, \dots, P_n\}$. There exists $\sigma_0 > 0$ and $\delta > 0$ such that for every $\sigma \geq \sigma_0$ there is a δ -hyperbolic space X , and a finite subset Φ of $G \setminus \{1\}$ with the following properties. For every collection of normal subgroups $N_i \trianglelefteq P_i$ with $N_i \cap \Phi = \emptyset$, $i = 1, 2, \dots, n$ there is a σ -rotating family \mathcal{R} where for every $(H, v) \in \mathcal{R}$, H is conjugate to some N_i . Moreover, if each N_i is of finite index in P_i , and K is the normal subgroup of G generated by $N_1 \cup \dots \cup N_n$, then $\bar{G} := G/K$ acts properly discontinuously and cocompactly on $\bar{X} := X/K$ which is $\bar{\delta}$ -hyperbolic for some $\bar{\delta} \leq 900\delta$. In particular, \bar{G} is an hyperbolic group.*

Remark. For sake of conciseness, we have preferred to state this version of [11, Proposition 7.7. and Corollary 7.8] because it is suitable for our applications. Roughly, to see how Proposition 4.2 follows from [11, Proposition 7.7.], one should start with the horoball definition of relatively hyperbolic group used in [11, Definition 7.1.] rather than Definition 4.1. From that, one uses the construction of [11, Lemma 7.2.] which roughly replaces the each horoball Y by a cone $Y \times [0, \sigma] / \sim$ (where \sim identifies all the points of the form $(y, 0)$), to produced a δ -hyperbolic space. It is important to note that δ depends on σ_0 but not on σ . The apices of these cones will be the apices of the rotation family and are stabilized by conjugates of the parabolic subgroups. The first part of the proposition is now implied by [11, Proposition 7.7.] It is worth noticing that the reason of the condition of avoiding a finite set Φ is to guarantee that the rotation groups “rotate with a large

angle". The moreover part follows from [11, Corollary 7.8] and the particular estimate for $\bar{\delta}$ follows from Theorem 3.2. An interested reader can check the details in [11, §7.1].

We now obtain an extension (Item (iii) of theorem below) of the Dehn fillings result for relatively hyperbolic groups, which was proved originally by Osin [21] and Groves and Manning [16].

Theorem 4.3. *Let G be finitely generated group hyperbolic relatively to a family of subgroups $\{P_1, \dots, P_n\}$. There is a finite set $\Phi \subseteq G \setminus \{1\}$ such that whenever we take finite index normal subgroups $N_i \trianglelefteq P_i$ with $N_i \cap \Phi = \emptyset$ for $i = 1, \dots, n$ then the following hold:*

- (i) $\bar{G} := G/K$ is an hyperbolic group where K is the normal subgroup of G generated by $N_1 \cup \dots \cup N_n$.
- (ii) there exists subsets T_i of G for $i = 1, \dots, n$ such that K is isomorphic to $*_{i=1}^n (*_{t \in T_i} N_i^t)$.
- (iii) for every $\bar{g} \in \bar{G}$ of infinite order, there is a pre-image g of \bar{g} under the natural map $G \rightarrow \bar{G}$ and subsets T'_i of G for $i = 1, \dots, n$ such that

$$\langle g, K \rangle = \langle g \rangle * \left[*_{i=1}^n (*_{t \in T'_i} N_i^t) \right].$$

Proof. It follows combining Proposition 4.2, Theorem 3.3 and Theorem 3.17. □

5 Farrell-Jones via Dehn fillings

Closure properties of Farrell-Jones groups Let \mathfrak{C} be the class of groups satisfying the K- and L-theoretic Farrell-Jones Conjecture with finite wreath products (with coefficients in additive categories) with respect to the family of virtually cyclic subgroups. The statement of the Farrell-Jones conjecture and its applications can be found in [4, 6]. We collect now some properties of the class \mathfrak{C} .

Proposition 5.1 ([6, 7, 4, 5][13, Proposition 4.1]). *The following properties hold:*

- (i) \mathfrak{C} is closed under taking subgroups.
- (ii) \mathfrak{C} is closed under free products.
- (iii) hyperbolic groups and abelian groups are in \mathfrak{C} .
- (iv) If $\pi: G \rightarrow \bar{G}$ is a morphism such that \bar{G} is in \mathfrak{C} and for every torsion-free cyclic subgroup \bar{H} of \bar{G} , $\pi^{-1}(\bar{H})$ is in \mathfrak{C} then G is in \mathfrak{C} .

Theorem 5.2. *Let G be a finitely generated group hyperbolic relative to residually finite groups in the class \mathfrak{C} . Then G is in the class \mathfrak{C} .*

Proof. Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be the peripheral subgroups of G . Let Φ be the finite subset given by Theorem 4.3. Recall that peripheral subgroups are residually finite. Hence for every $i \in \{1, \dots, n\}$, there exists a finite index normal subgroup N_i of P_i such that $N_i \cap \Phi = \emptyset$. Note that every N_i belongs to \mathfrak{C} (Proposition 5.1 (i)). Let K be the normal subgroup of G generated by $N_1 \cup \dots \cup N_n$. Applying Theorem 4.3 we get the following.

- (i) $\bar{G} := G/K$ is an hyperbolic group. In particular it belongs to \mathfrak{C} (Proposition 5.1 (iii)).

- (ii) There exists subsets T_i of G for $i = 1, \dots, n$ such that K is isomorphic to $*_{i=1}^n (*_{t \in T_i} N_i^t)$. As we noticed before every N_i is in \mathfrak{C} . According to [Proposition 5.1 \(ii\)](#) so is K .
- (iii) For every $\bar{g} \in \bar{G}$ of infinite order, there is a pre-image g of \bar{g} under the natural map $\pi: G \rightarrow \bar{G}$ and subsets T'_i of G for $i = 1, \dots, n$ such that

$$\langle g, K \rangle = \langle g \rangle * \left[*_{i=1}^n (*_{t \in T'_i} N_i^t) \right].$$

Recall that cyclic groups are in \mathfrak{C} . Applying again [Proposition 5.1 \(ii\)](#), we get that $\langle g, K \rangle$ is in \mathfrak{C} as well.

We apply [Proposition 5.1 \(iv\)](#) with $\pi: G \rightarrow \bar{G}$. Let \bar{H} be an torsion-free cyclic subgroup of \bar{G} . If \bar{H} is trivial, then we noticed that $K = \pi^{-1}(\bar{H})$ is in \mathfrak{C} . Otherwise, there exists a loxodromic element of \bar{G} generating \bar{H} . Then we observed just above that $K = \pi^{-1}(\bar{H})$ in a free product that lie again in \mathfrak{C} . \square

References

- [1] I. Agol. The virtual Haken conjecture. *Documenta Mathematica*, 18:1045–1087, 2013.
- [2] Y. Antolin, A. Minasyan, and A. Sisto. Commensurating endomorphisms of acylindrically hyperbolic groups and applications. *arXiv.org*, (to appear in Groups Geometry and Dynamics), Oct. 2013.
- [3] A. Bartels. Coarse flow spaces for relatively hyperbolic groups. *arXiv.org*, Feb. 2015.
- [4] A. Bartels, F. T. Farrell, and W. Lück. The Farrell-Jones conjecture for cocompact lattices in virtually connected Lie groups. *Journal of the American Mathematical Society*, 27(2):339–388, 2014.
- [5] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. *Annals of Mathematics. Second Series*, 175(2):631–689, 2012.
- [6] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Inventiones Mathematicae*, 172(1):29–70, 2008.
- [7] A. Bartels, W. Lück, H. Reich, and H. Rüping. K- and L-theory of group rings over $\mathrm{GL}_n(\mathbb{Z})$. *Publications Mathématiques. Institut de Hautes Études Scientifiques*, 119(1):97–125, 2014.
- [8] A. Bartels and H. Reich. Coefficients for the Farrell-Jones conjecture. *Advances in Mathematics*, 209(1):337–362, 2007.
- [9] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990.
- [10] R. Coulon. On the geometry of Burnside quotients of torsion free hyperbolic groups. *International Journal of Algebra and Computation*, 24(3):251–345, 2014.
- [11] F. Dahmani, V. Guirardel, and D. V. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *arXiv.org*, (1111.7048), Nov. 2011.

- [12] F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic K-theory. *Journal of the American Mathematical Society*, 6(2):249–297, 1993.
- [13] G. Gandini and H. Rüping. The Farrell-Jones conjecture for graph products. *Algebraic & Geometric Topology*, 13(6):3651–3660, 2013.
- [14] É. Ghys and P. de la Harpe. *Sur les groupes hyperboliques d’après Mikhael Gromov*, volume 83 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1990.
- [15] M. Gromov. CAT(κ)-spaces: construction and concentration. *Rossiiskaya Akademiya Nauk. Sankt-Peterburgskoe Otdelenie. Matematicheskii Institut im. V. A. Steklova. Zapiski Nauchnykh Seminarov (POMI)*, 280(Geom. i Topol. 7):100–140, 299–300, 2001.
- [16] D. Groves and J. Manning. Dehn filling in relatively hyperbolic groups. *Israel Journal of Mathematics*, 168:317–429, 2008.
- [17] G. C. Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. *Algebraic & Geometric Topology*, 10(3):1807–1856, 2010.
- [18] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, pages 269–322. Birkhäuser, Basel, Basel, 2005.
- [19] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, Berlin, Heidelberg, 2005.
- [20] D. V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Memoirs of the American Mathematical Society*, 179(843):vi–100, 2006.
- [21] D. V. Osin. Peripheral fillings of relatively hyperbolic groups. *Inventiones Mathematicae*, 167(2):295–326, 2007.
- [22] D. V. Osin and A. Thom. Normal generation and ℓ^2 -Betti numbers of groups. *Mathematische Annalen*, 355(4):1331–1347, 2013.
- [23] W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *American Mathematical Society. Bulletin. New Series*, 6(3):357–381, 1982.
- [24] C. Wegner. The K-theoretic Farrell-Jones conjecture for CAT(0)-groups. *Proceedings of the American Mathematical Society*, 140(3):779–793, 2012.
- [25] C. Wegner. The Farrell-Jones Conjecture for virtually solvable groups. *arXiv.org*, Aug. 2013.